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Comment on Noise and Bifurcations

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We calculate in exact form the first correction in a parameter measuring the strength of the noise to the effective potential for one-variable diffusion processes. The use of this potential to study transitions is discussed.

KEY WORDS: Bifurcations; effective potential; functional integration; noise; transitions.

In ref. 1 the effective potential was used to discuss transitions in a noisy dynamical system. In that interesting paper a critical analysis was done of the notion of a bifurcation point in a dynamical system in the presence of white noise. The conclusion was that the bifurcation point should be replaced more properly by a bifurcation region. One of the tools of the analysis was the effective potential and the arguments were illustrated with the normal form of the pitchfork bifurcation with additive white noise. The use of the effective potential for these problems, together with a stochastic interpretation of this function, was proposed by Graham[(2)] some time ago and applied to the Bénard problem. We shall calculate here exactly the first-order correction (in the intensity of the noise) to the effective potential for an arbitrary stochastic differential equation $\dot{q} = A(q) + \sqrt{n}\,\xi(t)$, where $\xi(t)$ is the noise. Then we specialize to the pitchfork case $A(q) = \mu q - q^3$ which was treated in ref. 1. We find no indication for the appearance of the bifurcation when $\mu < 0$, while for $\mu > 0$ one has two minima for $\eta/\mu^2 < 16/69$.

We consider the stochastic differential equation

$$\dot{q} = A(q) + \sqrt{n}\,\xi(t) \tag{1}$$

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with $\xi(t)$ a Gaussian white noise with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\, \xi(t') \rangle = \delta(t - t')$. The generating functional of correlation functions for the Markov process defined by (1) with deterministic initial condition $q(t_0) = \alpha_0$ is[3]

$$Z[J(\cdot)] = \int_{\gamma(0)} Dq\, D^{\eta}p \exp\left\{\frac{i}{\eta} \int_{t_0}^{T} dt \left[p(\dot{q} - A(q)) + \frac{i}{2} p^2 - iJq \right]\right\} \times \delta(q(t_0) - \alpha) \tag{2}$$

and one has ($\tau_j \in [t_0, T]$)

$$\prod_{i=1}^{m} \eta \frac{\delta}{\delta J(\tau_i)} Z[J] \bigg|_{J=0} = \langle q(\tau_1) \cdots q(\tau_m) \rangle \tag{3}$$

where $\gamma(0)$ stands for prepoint discretization and defines (2) as the limit when $N \to \infty$ of the multiple integral I_N,

$$I_N = \int \prod_{i=1}^{N+1} dq_i \prod_{j=1}^{N+1} \frac{dp_j}{2\pi\eta} \times \exp\left\{\frac{i\varepsilon}{\eta} \sum_{j=1}^{N+1} \left[p_j \left(\frac{\Delta q_j}{\varepsilon} - A(q_{j-1}) \right) + \frac{i}{2} p_j^2 - iJ(t_j)\, q_j \right]\right\} \tag{4}$$

where

$$\Delta q_j \equiv q_j - q_{j-1}, \qquad q_0 = \alpha_0, \qquad t_j = t_0 + j\varepsilon, \qquad t_{N+1} = T$$

$$\varepsilon = \frac{T - t_0}{N + 1}$$

We remark that in ref. 2 the Jacobian which appears in the derivation of this equation is absent, due to the prepoint discretization $\gamma(0)$ which we are using. In general the Jacobian term is[4]

$$\exp\left[-s \int_{t_0}^{T} dt\, A'(q) \right]$$

(the prime stands from now on for derivative) in the $\gamma(s)$ discretization which discretizes the q dependence in ref. 2 as $q(t) \to q_{j-1} + s\, \Delta q_j$ and for $s = 0$ it is just one. For additive noise all discretizations $\gamma(s)$ are equivalent and there is no reason for the choice $s = 1/2$ used in refs. 5 and 6 apart from the fact that integration by parts can be done with the usual formula in this case.[3,7] One can check in perturbation theory that the dependence in s disappears by cancellation at each order[8] (see also ref. 3). Putting

$$Z[J] = \exp\left(\frac{1}{\eta} W[J] \right)$$

one defines $\Gamma[q]$ as the Legendre transform of $W[J]$ by (see, for example, ref. 9, Chapter V)

$$\Gamma[q] + W[J] = \int dt\, J(t)\, q(t), \qquad \frac{\delta W}{\delta J(t)} = q(t) \tag{5}$$

From (3) one obtains

$$\left.\frac{\delta W}{\delta J(t)}\right|_{J=0} = \langle q(t) \rangle$$

It is simple to show from (2) that $W[J] = W^0[J] + \eta W^1[J] + O(\eta^2)$, and using this, one can obtain from (5) that $\Gamma[q] = \Gamma^0[q] + \eta\Gamma^1[q] + O(\eta^2)$. The stationary solution of (1) is obtained from (2) taking the limit $t_0 \to -\infty$ (since T is arbitrary, we take also $T \to \infty$); in this case one shows that $\Gamma[q]$ has the form

$$\begin{aligned} \Gamma^{\text{st}}[q] = \int_{-\infty}^{\infty} dt\, [&V(q(t)) + \dot{q}B_1(q(t)) + \dot{q}^2 B_2(q(t)) \\ &+ \ddot{q}B_3(q(t)) + \cdots] \end{aligned} \tag{6}$$

where $V(q)$ is the effective potential. This function can be evaluated from (6) if we put $q(t) = \alpha$; then

$$\Gamma^{\text{st}}[\alpha] = \int_{t_0}^{T} dt\, V(\alpha) = \tau V(\alpha), \qquad T - t_0 \equiv \tau \to \infty$$

so that if we can factorize in $\Gamma^{\text{st}}[q(\cdot) = \alpha]$, the integral $\int_{t_0}^{T} dt = \tau$ unambiguously, we can calculate the function $V(\alpha)$.

In order to calculate $W^0[J]$ and $W^1[J]$ we proceed in the usual way.[9]

The integrand in (2) can be written as $(p\dot{q} - H^J(p, q))$ with

$$H^J = pA(q) - \frac{i}{2}p^2 + iJq$$

Let $(q = u^J(t),\ p = v^J(t))$ be solutions of the Hamilton equations for H^J, which are

$$\dot{q} = \frac{\partial H^J}{\partial p} = A(q) - ip, \qquad \dot{p} = -\frac{\partial H^J}{\partial q} = -pA'(q) - iJ \tag{7}$$

We choose $u^J(t) = u(t) + O(J)$, $v^J(T) = O(J)$, i.e., for $J = 0$ one has $v^{J=0}(t) = 0$, $u^{J=0}(t) = u(t)$ with $u(t)$ the solution of the deterministic equation $\dot{u} = A(u(t))$ with initial condition $u(t_0) = \alpha_0$. We make now in (2) the change of variables

$$q(t) = u^J(t) + \sqrt{\eta}\, Q(t), \qquad p(t) = v^J(t) + \sqrt{\eta}\, P(t) \tag{8}$$

which gives {after replacing $v^J = i[\dot{u}^J - A(u^J)]$ from (7)}

$$Z[J] = \exp\left(-\frac{1}{\eta}\int_{t_0}^{T} dt \left\{\frac{1}{2}[\dot{u}^J - A(u^J)]^2 - Ju^J\right\}\right) \times G[u^J][1 + O(\eta)] \tag{9}$$

$$G[u^J] = \int_{\gamma(0)} DQ\, DP \left(\exp i \int_{t_0}^{T} dt \left\{P(\dot{Q} - A'(u_J)Q) + \frac{i}{2}P^2 + \frac{1}{2}[\dot{u}^J - A(u^J)]\, A''(u^J)\, Q^2\right\}\right) \delta(Q(t_0)) \tag{10}$$

where the discretized version of $DQ\, DP$ is

$$\prod_{i=1}^{N+1} dQ_i \prod_{j=1}^{N+1} \frac{dP_j}{2\pi}$$

thus showing that $G[u^J]$ does not depend on η. Notice that the linear terms in (Q, P) vanish since (u^J, v^J) are solutions of the Hamilton equations for H^J.

From these formulas we obtain

$$W^0[J] = -\mathscr{A}[u^J] + \int dt\, Ju^J, \qquad W^1[J] = \ln G[u^J] \tag{11}$$

with

$$\mathscr{A}[q] = \int dt\, L(q, q), \qquad L = \tfrac{1}{2}[\dot{q} - A(q)]^2$$

One has $\delta W^0/\delta J(t) = u^J(t)$ and then we see from (5) that at lowest order $q(t) = u^J(t)$. The functional $\Gamma[q]$ up to first order in η can be calculated from (5) as

$$\Gamma[q] = -W^0[J] - \eta W^1[J] + \int dt\, J(t)\, q(t) + O(\eta^2) \tag{12}$$

where J must be replaced as a functional of q which is obtained by inversion of

$$\frac{\delta}{\delta J(t)}(W^0 + \eta W^1) = q(t)$$

But in fact one can easily see that in (12) one can still use the lowest-order result $q(t)=u^J(t)$. Then, using (11), we have

$$\Gamma[q]=\mathscr{A}[q]-\eta\ln G[q]+O(\eta^2) \tag{12}$$

where the functional $G[q]$ is given by (10).

Since we are finally interested in $V(q)$ appearing in $\Gamma^{\rm st}[q]$ [see (6)], we put in (10) $u^J(t)=q=\text{const}$ and it must be understood that in order to compare with (6) one must take the limit $t_0\to-\infty$, $T\to\infty$. After doing the Gaussian integration DP in (10) we obtain (q is a constant now)

$$\begin{aligned}G^{\rm st}[q]=\int_{\gamma(0)}\prod_{i=1}^{N+1}\frac{dQ_i}{\sqrt{2\pi\varepsilon}}\\ &\times\exp\left(-\frac{1}{2}\int_{t_0}^{T}dt\,\{[\dot{Q}-A'(q)Q]^2+A(q)\,A''(q)Q^2\}\right)\\ &\times\delta(Q(t_0))\end{aligned} \tag{13}$$

where $\delta(Q(t_0))$ just means that in the prepoint discretized version of (13) which we are using one must put $Q_0=0$.

We can write $G^{\rm st}[q]=\int d\bar{Q}\,P(\bar{Q},T\mid 0,t_0)$ with

$$\begin{aligned}P(\bar{Q},T\mid 0,t_0)=\frac{1}{\sqrt{2\pi\varepsilon}}\int_{\gamma(0)}\prod_{i=1}^{N}\frac{dQ_i}{\sqrt{2\pi\varepsilon}}\\ &\times\exp\left\{-\frac{1}{2}\int_{t_0}^{T}dt\,[(\dot{Q}-A'Q)^2+AA''Q^2]\right\}\\ &\times\delta(Q(t_0))\,\delta(Q(T)-\bar{Q})\end{aligned} \tag{14}$$

where the δ-functions mean that $Q_0=0$, $Q_{N+1}=\bar{Q}$ in the discretized version. In the argument of the exponential we have the term $\int dt\,\dot{Q}Q$ discretized in the prepoint, but one has[3,7]

$$\int_{\gamma(0)}dt\,Q\dot{Q}=\int_{\gamma(1/2)}dt\,Q\dot{Q}-\tfrac{1}{2}(T-t_0)=\tfrac{1}{2}[\bar{Q}^2-(T-t_0)] \tag{15}$$

since we can integrate by parts in the midpoint $\gamma(1/2)$ discretization. Then

$$P=\exp\left[\frac{A'(q)}{2}(\bar{Q}^2-\tau)\right]\cdot K(\bar{Q},T\mid 0,t_0)$$

with

$$K(\bar{Q},T\mid 0,t_0)=\frac{1}{\sqrt{2\pi\varepsilon}}\int_{Q_0=0}^{Q_{N+1}=\bar{Q}}\prod_{i=1}^{N}\frac{dQ_i}{\sqrt{2\pi\varepsilon}}\exp\left[-\frac{1}{2}\int_{t_0}^{T}dt\,(\dot{Q}^2+\lambda^2Q^2)\right] \tag{16}$$

where we have omitted $\gamma(0)$, since now one can discretize $Q(t)$ in an arbitrary way and $\lambda^2 = A'(q)^2 + A(q)\,A''(q)$. Putting $\bar{L}(Q, Q) = \frac{1}{2}(\dot{Q}^2 + \lambda^2 Q^2)$, we assume now that $\lambda(q)^2 > 0$ (the values of q for which this is not valid do not allow us to factorize $\int_{t_0}^{T} dt = \tau$ at the end of the calculation) and we make in (16) the displacement $Q(t) \to \alpha(t) + Q(t)$ with $\alpha(t)$ the solution of the Euler–Lagrange equations for $\bar{L}$ with boundary conditions $\alpha(t_0) = 0$, $\alpha(T) = \bar{Q}$. Then

$$K(\bar{Q}, T \mid 0, t_0) = \exp[-\tfrac{1}{2}\dot{\alpha}(T)\,\alpha(t)] \cdot K(0, T \mid 0, t_0) \tag{17}$$

since $\int dt\,\bar{L}(\alpha, \dot{\alpha}) = \frac{1}{2}\dot{\alpha}(T)\,\alpha(T)$. On the other hand (ref. 3, Chapter IX), $K(0, T \mid 0, t_0) = [2\pi D(t_0)]^{-1/2}$, where $D(t)$ satisfies the same equation as $\alpha(t)$ but with boundary conditions $D(t) = 0$, $\dot{D}(T) = -1$.

Putting all this together and doing the Gaussian integral over $\bar{Q}$, one finally obtains

$$G^{\text{st}}\,|q| = \exp\left(-\frac{A'(q)}{2}\tau\right) \times \left(\frac{2\lambda}{(\lambda - A')\exp(\lambda\tau) + (\lambda + A')\exp(-\lambda\tau)}\right)^{1/2} \tag{18}$$

In the limit $\tau \equiv T - t_0 \to \infty$ one has

$$\ln G^{\text{st}}[q] = -\frac{\tau}{2}[A'(q) + \lambda(q)]$$

and consequently the effective potential obtained from (12) has the value

$$V(q) = \frac{1}{2}A(q)^2 + \frac{\eta}{2}A'(q) + [A'(q)^2 + A(q)\,A''(q)]^{1/2} \tag{19}$$

From (5) one has $\delta\Gamma/\delta q(t) = J(t)$ and since

$$\left.\frac{\delta W}{\delta J(t)}\right|_{J=0} = \langle q(t) \rangle$$

we have

$$\left.\frac{\delta\Gamma}{\delta q(t)}\right|_{q(t) = \langle q(t) \rangle} = 0$$

and also $\Gamma[q(t) = \langle q(t) \rangle] = 0$. In the stationary case $\langle q(t) \rangle^{\text{st}} = v = \text{const}$ and from (6) we obtain $V'(q = v) = 0$ and $V(q = v) = 0$. Moreover, one must also have[2] $V''(q = v) > 0$ since

$$\left.\frac{\delta^2 \Gamma^{\text{st}}}{\delta q(t)\,\delta q(t')}\right|_{q=v}$$

is just the inverse of the stationary correlation function. One should remark that if one has more than one locally stable equilibrium point for $\dot{q} = A(q)$, i.e., $A(q) = 0$ has for example, two solutions $\bar{q}_1$ and $\bar{q}_2$ such that $A(\bar{q}_l) = 0$, $A'(\bar{q}_l) < 0$, $l = 1, 2$, then one can construct, at least as perturbation series in powers of η, two systems of stationary correlation functions $\langle q(\tau_1) \cdots q(\tau_m) \rangle^{(l)}$ and this from two generating functionals $Z_l^{\text{st}}[J]$. From this we expect that $V(q)$, which is the same function in both cases, should have two local minima at $q = v_l$, $v_l = \bar{q}_l + O(\eta)$, $l = 1, 2$. This does not mean that $\langle q(t) \rangle^{\text{st}}$ equals one of these values, but rather that the system goes to one of these states and has an exponentially vanishing probability when $\eta \to 0$ of the form $\exp(-a/\eta)$ of going out to the other state.

Let us specialize now to $A(q) = \mu q - q^3$, which is the normal form of the pitchfork bifurcation. From (19) we have

$$\begin{aligned} V(q) &= \frac{(\mu q - q^3)^2}{2} + \frac{\eta}{2} \{\mu - 3q^2 + [f(q)]^{1/2}\} \\ f(q) &= \mu^2 - 12\mu q^2 + 15 q^4 \end{aligned} \tag{20}$$

We shall distinguish two cases: (a) $\mu < 0$ and (b) $\mu > 0$.

In case (a), $f(q) > 0$ always and (20) defines a function $V_1(q)$ for all q. This function has only one minimum at $q = 0$ and there $V_1(0) = V_1'(0) = 0$, $V_1''(0) = \mu^2/2 + \frac{3}{2}\eta > 0$. This last number can be checked by the usual perturbation theory around $q = 0$ (it is enough to calculate the first correction to the correlation function). For $\mu = 0$ the potential reduces to $\tilde{V}(q) = \frac{1}{2} q^6 + \frac{1}{2}\eta q^2(\sqrt{15} - 3)$, $\tilde{V}(0) = \tilde{V}'(0) = 0$, $\tilde{V}''(0) = \eta(\sqrt{15} - 3) > 0$.

In case (b), $f(q) > 0$ only for q^2 outside an interval $[a_1, b_1]$, $a_1 \cong 0.1\mu$, $b_1 \cong 0.7\mu$, and (20) defines the potential only outside this interval. For $\mu > 0$ one has bistability since the deterministic problem has two locally stable states $\bar{q}_{1,2} = \pm\sqrt{\mu}$ for which $A(\bar{q}_l) = 0$ and $A'(\bar{q}_l) < 0$, and at the same time the state $q = 0$ becomes unstable since $A'(0) = \mu > 0$. One can also show that the mean values in the stationary states are now $\langle q(t) \rangle = v_{12} = \pm v$, $v = \sqrt{\mu}\,(1 - \frac{3}{8}\eta/\mu^2)$. From (20) we obtain now a function $V_2(q)$ which has three local minima at $q = 0$ and $q = v_l$, $l = 1, 2$, and

$$V_2(v_l) = V_2'(v_l) = 0, \qquad V_2''(v_l) = 4\mu^2 \left(1 - \frac{69}{16}\frac{\eta}{\mu^2}\right)$$

Again this value can be checked by perturbation theory around $q = v_l$ now. However, one has now that $V_2(0) = \eta\mu > 0$, $V_2'(0) = 0$, $V_2''(0) = \mu^2(1 - 9\eta/\mu^2)$, i.e., $V_2(q)$ has its absolute minimum at $q = \pm v$ since $V_2(0) = \eta\mu > V_2(\pm v) = 0$. We remark that $V_2''(v_l)$ becomes negative at $\sigma = \eta/\mu^2 > \sigma_c = 16/69$, where σ is the effective expansion parameter in this model, since an appropriate scaling puts the initial equation in the form $\dot{q} = q - q^3 + \sqrt{\sigma}\,\xi(t)$. But at $\sigma = \sigma_c$ we enter the critical region[10] where the two systems of stationary correlation functions $\langle q(\tau_1) \cdots q(\tau_m) \rangle^{(l)}$ with $\langle q(\tau) \rangle^{(l)} = v_l$ lose their meaning, since the time scale determined by the Arrhenius–Boltzmann factor is of order one, i.e., the escape time from v_l is of order one. We can then say that the effective potential up to first order in σ gives information about the critical bifurcation region.

In summary, the potential $V(q)$ calculated up to $O(\sigma)$ has for $\mu < 0$ only one minimum and for $\mu > 0$ two minima for $\sigma < \sigma_c$.

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